— 2015 —

Lecture 1

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Plan

- basic notions about fixed income securities
- A quick look at the no arbitrage theory and change of numeraire technique
- No arbitrage theory for bond Market
- Bond market based on short rate models
- HJM framework for Bond Markets
- Market models.

Def (Fixed income securities). Securities which promises the investor specific payments at specific future dates.

They are also called debt instruments. One of the fundamental debt instrument is BOND.

A bond is a contract which gives the holder (promise) payments at regular

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interval of time, (periodical) and a final payment at the maturity date.

Regular payments in between are called coupons and final payment is called the principal/face value.

In a domestic bond market, bonds are issued by various market participants such as

- Government bonds: issued by central governments, eg. treasury bills/bonds
- Guaranteed bond: issued by government agencies, for example bond by Railways.
- State and local bonds: issued by local governments, eg. municipal bonds.
- Corporate bonds: issued by private companies.

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Interest rates. Interest rate is the amount of money a borrower promises to pay the lender. Hence all fixed income securities have some underlying ~~rates~~ (interest rates (called the internal rate of return / yield)

Examples (1) treasury rates, the rates an investor earns from a treasury bill / bond. These ~~instr~~ instruments are ~~risk f~~ default free. Hence some times these interest rates are substitutes for the risk-free rates. (Short rate)

(2) LIBOR rates: The rate at which a bank is prepared

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to make a deposit with other banks.

Like treasury bills, LIBOR rates exist for 1, 3, 6 and 12 month maturities.

~~LIBOR rates~~ Before seting into the yields of bonds, we look at classification of bonds

One can classify bonds according to the structure.

- Fixed coupon bonds: standard coupon bonds. coupon rates will be specied. For example: 1-year, 6% semi-an coupon bond ; ~~means~~ means - - - .
- Zero coupon bond: pay no coupon. (Though usually no longer maturity zero coupon bonds are not traded, it is important for analysis)

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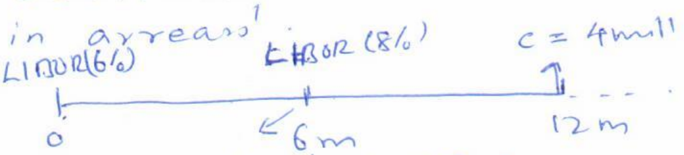
- Floating rate notes (FRN): pays coupons equal to a reference (floating) rate + a margin and pays face value at maturity.

Eg: '10 million FRN paying semi-annual 6-month LIBOR in arrears'



$c = \frac{1}{2} \times 100 \times 6 = 3$mill

- Structured notes: have more complex payments to satisfy investor's needs.

Eg: Inverse floaters have coupon payments that vary 'inversely' with the level of interest rates. A typical

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coupon rate is $c = \max\{12\% - \text{LIBOR}, 0\}$.

- Callable bonds: issuer has the right to call back the bond at fixed prices on fixed dates.
- Puttable bonds: investor has the right to put back the bond to the issuer at fixed rates on fixed dates.
- Convertable bonds, where the bond can be converted into the stock of the company at a fixed date at a fixed price.

As we have seen earlier that there ~~is a short rate~~ the market has a risk-free rate process called the short rate process.

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This gives the Money market account / Bank account.

If $B(t)$ denote the value at time t of 1 unit of money invested at time 0 in the money market, then

$$\left.\begin{aligned} B(t) &= e^{\int_0^t r(s)\,ds}, \text{ OR} \\ dB_t &= r(t) B_t\,dt, \; B_0 = 1, \end{aligned}\right\} (1)$$

where $r(\cdot)$ is a process (with continuous paths....).

In (1), one can associate the discount process

$$D(t,T) = e^{-\int_t^T r(s)\,ds},$$

the value at t of 1 unit in the future T.

- Zero Coupon (8) Bond

Recall that zero-coupon bond promise the holder the face value at maturity but no coupons in between.

The contract value at t of a T-zero bond denoted by $P(t,T)$ is called the bond price.

$P(t,T) \equiv$ the present (t) value of 1 unit cash of the future T.

Assume that we are dealing with default free bonds!

Question 1: Is $P(t,T) = D(t,T)$?

Answer: In general No. Yes if $r(\cdot)$ is deterministic.

Question 2: What is the yield?
To answer this we need to specify how one calculate internal rate of return.
There are various practices
I Day count convention
II type of compounding.
I: time to matury:
$\tau(t,T)$ = time (in years) from present (t) to future T.
Let $t = (d_1, m_1, y_1)$
$T = (d_2, m_2, y_2)$.
Then $\tau(t,T) = T-t$ if $T-t \le 1$ day
For $T-t > 1$ day.
Various market practices
• Actual/365 $\equiv \frac{\text{Actual no. days}}{365}$
(Convention 1 year = 365 days)

• Actual/360 $\equiv \frac{\text{Actual no. days}}{360}$

• 30/360 $\equiv$
$$\frac{1}{360}\left[360(y_2-y_1)+30(m_2-m_1-1) + \max\{30-d_1,0\}+\min\{30,d_2\}\right]$$
(months $\equiv$ 30 days, year $\equiv$ 360 days)
• Modifications involving / excluding holidays.

Now onwards we denote $\tau(t,T)$ for the time to maturity.

II Different types of compounding given different types of yields. Start from here lecture
• Simple compounded spot rate: Investment accrues

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at a constant rate $L(t,T)$ during the time $\tau(t,T)$.
i.e
$$\big(1+\tau(t,T)L(t,T)\big)P(t,T)=1$$
$$\Rightarrow L(t,T) = \frac{1-P(t,T)}{\tau(t,T)P(t,T)} \quad (1.1)$$

LIBOR rates are simple compounded and Actual/360 convention is used.

• Annual compound spot rate is given by
$$(1+Y(t,T))^{\tau(t,T)} P(t,T) = 1$$
$$\Rightarrow Y(t,T) = \frac{1}{P(t,T)^{\frac{1}{\tau(t,T)}}} - 1$$

• n-times compound spot rate
$$\left(1+\frac{Y^n(t,T)}{n}\right)^{n\tau(t,T)} P(t,T) = 1$$

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$$\Rightarrow Y^n(t,T) = \frac{n}{P(t,T)^{\frac{1}{n\tau(t,T)}}} - n.$$

• Continuous compounding spot rate given by
$$e^{\tau(t,T)R(t,T)} P(t,T) = 1$$
$$\Rightarrow R(t,T) = -\frac{\ln P(t,T)}{\tau(t,T)} \quad (1.2)$$

Thus, we have defined various yield formulas for the T-bond. This leads to various family of curves.

Interest rate curves
• Zero coupon curves
$$(1) \quad T \mapsto \begin{cases} L(t,T) & t<T\le t+1 \\ Y(t,T) & T>t+1 \end{cases}$$

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(2) $T \longmapsto R(t,T)$

- Zero bond curve

$T \longmapsto p(t,T)$.

Now we discuss the so called forward rates:

Forward rates are interest rates 'locked' today for an ~~rates~~ investment in a future time period $[T, S]$.

To do this, we need another instrument called forward rate agreements.

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Def (FRA). Forward rate agreement with maturity S is a contract in which the holder receives a payment based on a fixed rate K and makes payments based on the floating rate $L(T,S)$ for the nominal N.

Contract value.

Payoff at maturity S

$$C(\text{Holder}) = N\tau(T,S)(K - L(T,S)).$$

Substitution

$$L(T,S) = \frac{1 - P(T,S)}{\tau(T,S)P(T,S)}$$

we get (do the substitution)

$$\text{Payoff} = N\left[(\tau(T,S)K+1) - \frac{1}{P(T,S)}\right]$$

~~First~~ we calculate the value of the payoff.

~~Consider~~

$$(\tau(T,S)K+1)(P(T,S)).$$

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Use S-bonds

Value of $(\tau(T,S)K+1)$ S-bonds at $t = (\tau(T,S)K+1)P(t,S)$

Value of $\frac{1}{P(T,S)}$ at time t

- 1 T-bond at time t and use the face value of T-bond to purchase $\frac{1}{P(T,S)}$ S-bond at time T.

This leads to payoff $\frac{1}{P(T,S)}$ at S

Initial value of the investment at t $= P(t,T)$.

$\therefore$ value of $\frac{1}{P(T,S)}$ at $t = P(t,T)$

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Combining these we get

$$FRA(t,T,S,\tau(T,S),K,N) = N\left[(\tau(T,S)K+1)P(t,S) - P(t,T)\right] \quad \text{--- (1.3)}$$

Forward rate interest

The fair value of K in the FRA is called the forward rate of interest.

i.e K satisfying

$$FRA(t,T,S,\tau(T,S),K,N) = 0.$$

$$\Rightarrow (\tau(T,S)K+1)P(t,S) = P(t,T)$$

$$\Rightarrow F(t;T,S) = \left[\frac{P(t,T)}{P(t,S)} - 1\right]\frac{1}{\tau(T,S)} \quad (1.4)$$

i.e ~~$F(t,T,S) =$~~

One can rewrite the FRA value using $F(t;T,S)$ as follows.

$FRA(t,T,S,\tau(T,S),K,N)$

$= N[(\tau(T,S)K+1)P(t,S)-P(t,T)]$

$= N[\tau(T,S)K P(t,S) - (P(t,T)-P(t,S))]$

$= N\tau(T,S)P(t,S)\left[K - \frac{P(t,T)-P(t,S)}{\tau(T,S)P(t,S)}\right]$

$= N\tau(T,S)P(t,S)\left[K - F(t;T,S)\right]$ ..... (1.5)

i.e Value of FRA is the discounted value of payoff replacing $L(T,S)$ in the payoff with $F(t;T,S)$.

i.e $F(t;T,S) \equiv$ expectation about $L(T,S)$ today!

What is $\lim_{S\downarrow T} F(t;T,S)$

i.e. instantaneous forward rate.

If $P(t,\cdot)$ is smooth, then

$$f(t,T) = \lim_{S\downarrow T} - \frac{P(t,S)-P(t,T)}{P(T,S)(S-T)}$$

$$= -\frac{\partial \ln P(t,T)}{\partial T} \quad \dots (1.6)$$

and

$r(t) = f(t,t)$ is called the short rate.

~~Swap Rate~~ Interest Rate Swap.

In any swap contract, parties involved are

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allowed to interchange their cash flow streams.

Fixed leg pays $N\tau_i K$, $i=1,2,\dots,n$ at T_i, where

(i) $T_0, T_1, \dots, T_n$ are specific set of dates

(ii) N is the nominal (take as 1)

(iii) K is a fixed rate.

(iv) $\tau_i \stackrel{def}{=} \tau(T_{i-1},T_i)$,

Floating leg pays $N\tau_i L(T_{i-1},T_i)$ at T_i, $i=1,2,\dots,n$.

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Receiver IRS receives fixed leg and pays floating leg.

Valuation of Receiver IRS

$$t \quad T_0 \quad T_1 \quad T_2 \quad \cdots \quad T_n$$

Fixed leg $\equiv (K\tau_1 N, \dots, K\tau_n N)$

Floating leg : $(K\tau_1 L(T_0,T_1), \dots, K\tau_n L(T_{n-1},T_n))$

Cashflow stream of Receiver IRS is

$(N\tau_1(K-L(T_0,T_1)), \dots, N\tau_n(K-L(T_{n-1},T_n)))$

An IRS swaps fixed leg payment with a floating leg payment.

For example Payer IRS pay ~~receives~~ fixed leg and receives float leg.

$\approx$ Combination of FRAs with maturities $T_1, T_2, \dots, T_n$.

$\therefore$ Value at t

$RFS(t,\{T_1,\dots,T_n\},K,N)$

$= \sum_{i=1}^{n} FRA(t,T_{i-1},T_i,\tau_i,K,N)$.

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Recall that

$FRA(t; T_{i-1}, T_i, \tau_i, K, N) = N\tau_i P(t,T_i) \times (K - F(t;T_{i-1},T_i))$

Hence

$RFS(t; \{T_1,..,T_n\}, K, N)$

$$= N \sum_{i=1}^{n} \tau_i P(t,T_i)(K - F(t;T_{i-1},T_i)) \quad , t \le T_0 \quad \text{------ (1.7)}$$

Def (Forward Swap rate)

Forward swap rate is defined as the fair value of K.

i.e value of K for which

$RFS(t; \{T_1,..,T_n\}, K, N) = 0$.

From (1.7) we set

$$\sum_{i=1}^{n} \tau_i P(t,T_i)(K - F(t;T_{i-1},T_i)) = 0$$

Recall that $F(t;T,S) = \dfrac{P(t,T) - P(t,S)}{\tau(T,S)P(t,S)}$

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Hence

$\tau_i P(t,T_i) \times F(t;T_{i-1},T_i)$.

$= P(t,T_{i-1}) - P(t,T_i)$.

$$\therefore K \sum_{i=1}^{n} \tau_i P(t,T_i) = \sum_{i=1}^{n} \tau_i P(t,T_i) F(t;T_{i-1},T_i)$$

$$= \sum_{i=1}^{n} (P(t,T_{i-1}) - P(t,T_i))$$

Hence

$$K \sum_{i=1}^{n} \tau_i P(t,T_i) = P(t,T_0) - P(t,T_n)$$

$$\text{i.e } K = \frac{P(t,T_0) - P(t,T_n)}{\sum_{i=1}^{n} \tau_i P(t,T_i)} \quad \text{.. (1.8)}$$

We denote Forward Swap rate by $S_{T_1,T_2}(t)$, $t \le T_0$.

Now we start with the no-arbitrage theory.

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A <u>quick look at stochastic calculus for continuous</u> ~~continuous~~ <u>semi martingales.</u>

$(\Omega, \mathcal{F}, Q_0)$ is a ~~complete~~ probability space and $\{\mathcal{F}_t\}$ is a filtration satisfying the usual conditions. i.e

(i) $\mathcal{F}_0$ contains all Q_0-null sets

(ii) $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$, $0 \le t < T$,

Here $T > 0$ a fixed time, we only observe the market upto this.

. A continuous super martingale $X = \{X_t \mid 0 \le t \le T\}$ is a

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process with continuous paths satisfying

(i) X is $\{\mathcal{F}_t\}$ adapted and each X_t is integrable ~~ans~~.

(ii) $E[X_t \mid \mathcal{F}_s] \le X_s$, $s \le t$.

X is a <u>continuous martingale</u> if both $X, -X$ are continuous super martingales.

$$\boxed{\text{A +ve process } X \text{ is a } Q_0 \text{ martingale iff it is a super martingale and } E^{Q_0} X_T = E^{Q_0} X_0} \quad \text{(I)}$$

An <u>adapted</u> continuous process $\{X_t \mid 0 \le t \le T\}$ is a <u>local martingale</u> if $\exists\, \tau_n \uparrow T$ such that $\{X_{t\wedge\tau_n} \mid t \ge 0\}$ is a M $\forall n \ge 1$.

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(II)

$\boxed{\text{Every positive local martingale is a super martingale / use conditional Fatou}}$

(I) & (II) ⇒

(III)

$\boxed{\text{Every +ve local martingale is a martingale} \iff EX_T = EX_0}$

Def. A process X is said to be continuous semi martingale if X has the following decomposition

$$X_t = A_t + M_t, \quad 0 \le t \le T$$

where $\{A_t\}$ is a continuous adapted ~~adp~~ process of BV and $\{M_t\}$ is a continuous local martingale.

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· Predictable process

A process $H = \{H_t \mid 0 \le t \le T\}$ is said to be predictable w.r.t $\{\mathcal{F}_t\}$ if the map $H : [0,T] \times \Omega \to \mathbb{R}^d$ is measurable w.r.to the predictable σ-field, which is the σ-field generated by all processes of the form

$$\sum_{i=0}^{n-1} X_i \, I_{(t_i, t_{i+1}]}, \text{ where}$$

X_i is $\mathcal{F}_{t_i}$ m'ble bounded random variable.

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Examples 1. All lcrl processes are predictable processes.

2. $X_t = I_{[1,\infty)}(t)$ is a predictable process (Ex)

Note this doesn't have ~~rcce~~ lcrl paths.

3. Let τ be a predictable stopping time, i.e, $\exists\, \tau_n \nearrow \tau$, where τ_n is a sequence of stopping times.

Then $X_t \overset{\text{def}}{=} I_{\{\tau \le t\}} = I_{[\tau,\infty)}(t), \; t \ge 0$

is a predictable process.

Define $X^n_t = I_{(\tau_n, \infty)}(t)$

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Then X^n is lcrl and hence predictable.

$\therefore \sigma(X^n) \in \mathcal{P}$, the predictable σ-field.

Now ~~$[\tau,\infty) = \bigcap_n (\tau_n,\infty) \in \mathcal{P}$~~

$$X^{-1}\{1\} = \{(\omega,t) / t \in [\tau(\omega),\infty)\}$$

$$= \bigcap_n \{(\omega,t) \mid t \in (\tau_n(\omega),\infty)\}$$

$$= \bigcap_n X^{n\,-1}\{1\} \in \mathcal{P}$$

$\therefore \sigma(X) \in \mathcal{P}$

⇒ X is predictable.

A process H is locally bounded if $\exists$ stopping times $\tau_n \nearrow T$ such that

$$\sup_{0 \le t \le T} |H_{t \wedge \tau_n}| < \infty \text{ a.s } \forall n \ge 1$$

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For a local continuous martingale M and a predictable locally bounded process H we define $\int_0^t H_s\,dM_s$ in the following steps.

Step I Once $\int_0^t H_s\,dM_s$ is defined for bdd predictable and continuous martingale.

Then define

$$\int_0^t H_s\,dM_s = \lim_{n\to\infty} \int_0^t H_{s\wedge\tau_n}\,dM_{s\wedge\tau_n}$$

Step II Define $\int H\,dM$ when H is simple predictable

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and approximate using L^2-isometry.

For a continuous semi-martingale X

$$\int_0^t H_s\,dX_s \overset{\text{def}}{=} \underbrace{\int_0^t H_s\,dA_s}_{\text{usual pathwise}} + \int_0^t H_s\,dM_s$$

$$\boxed{\int_0^{\cdot} H_s\,dM_s \text{ is a local martingale}}$$

H local bdd, predictable. M is local martingale.

For two continuous semi-martingales X, Y

$$\langle X, Y\rangle_t = X_t Y_t - \int_0^t X_s\,dY_s - \int_0^t Y_s\,dX_s.$$

called the cross-variation process.

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- $\{[X, Y]_t\}_{t\ge 0}$ is a process of bounded variation

$$\boxed{\text{If } X \text{ OR } Y \text{ is of bdd variation, then } \langle X, Y\rangle_t \equiv 0.}$$

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Change of Numéraire

No need of Money market account. In fact each T-claim should have their own suitable choice for discounting.

Def Any +ve $\{\mathcal{F}_t\}$-adapted process $Z = \{Z_t \mid 0 \le t \le T\}$ of the form
$$Z_t = Z_0 + \int_0^t \varphi_s \cdot dS_s$$
for some admissible φ is said to be a numéraire.

Theorem 3.2 Let $\{Z_t \mid 0 \le t \le T\}$ be a numéraire. Then φ is self-financing $\iff$
$$Z_t^{-1} V_t(\varphi) = Z_0^{-1} V_0(\varphi) + \int_0^t \varphi_u \cdot d(Z_u^{-1} S_u), \quad 0 \le t \le T.$$

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Hence
$$d(Z_t^{-1} V_t(\varphi)) = \varphi_t \cdot d(Z_t^{-1} S_t).$$
Reverse is similar ▨

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Theorem 3.3: Suppose there exists a numéraire Z and an equivalent probability measure $\mathbb{Q}^Z$ on $(\Omega, \mathcal{F})$ such that $\left\{ \frac{S_t}{Z_t} \mid 0 \le t \le T \right\}$ is a martingale under $\mathbb{Q}^Z$. If Y is another numéraire such that $E^{\mathbb{Q}^Z}\left[\frac{Y_T}{Z_T}\right] = E\left[\frac{Y_0}{Z_0}\right]$ then there exists a probability measure $\mathbb{Q}^Y \approx \mathbb{Q}^0$ such that $\left\{ \frac{S_t}{Y_t} \mid 0 \le t \le T \right\}$ is a $\mathbb{Q}^Y$-martingale.

Proof suppose φ is self-financing. Using product rule
$$d(Z_t^{-1} V_t(\varphi)) = V_t(\varphi)\, d(Z_t^{-1}) + Z_t^{-1} dV_t(\varphi) + d\langle Z^{-1}, V(\varphi)\rangle_t \quad (1)$$
consider self-fin
$$\langle Z^{-1}, V(\varphi)\rangle_t = \sum_{i=0}^n \left\langle Z^{-1}, \int_0^t \varphi_u^i dS_u^i \right\rangle_t$$
$$= \sum_{i=0}^n \int_0^t \varphi_u^i \, d\langle Z^{-1} S^i\rangle_u$$
$$= \int_0^t \varphi_u \cdot d\langle Z^{-1}, S\rangle_u \quad (2)$$
Hence (1) & (2) $\Rightarrow$
$$d(Z_t^{-1} V_t(\varphi)) = \cancel{V_t(\varphi)}\, \varphi_t \cdot S_t \, d(Z_t^{-1}) + Z_t^{-1} \varphi_t \cdot dS_t + \varphi_t \cdot d\langle Z^{-1}, S\rangle_t$$
$$= \varphi_t \cdot \left[S_t d(Z_t^{-1}) + Z_t^{-1} dS_t + d\langle Z^{-1}, S\rangle_t \right]$$

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Proof There exists φ admissible such that
$$Y_t = V_t(\varphi)$$
$\therefore$ By Theorem 3.2
$$Z_t^{-1} Y_t = Z_0^{-1} Y_0 + \int_0^t \varphi_u \cdot d(Z_u^{-1} S_u)$$
Now from ~~the defn~~ the property of stochastic integrals $\{Z_t^{-1} Y_t \mid 0 \le t \le T\}$ is a +ve local martingale under $\mathbb{Q}^Z$.
Since $E^{\mathbb{Q}^Z}\left[\frac{Y_T}{Z_T}\right] = E^{\mathbb{Q}^Z}\left[\frac{Y_0}{Z_0}\right]$, it follows that $\{Z_t^{-1} Y_t \mid 0 \le t \le T\}$ is a $\mathbb{Q}^Z$-martingale.

(#3)

Hence $E^{\mathbb{Q}^Z}\left[\frac{Y_T Z_0}{Z_T Y_0}\right] = 1$ $\quad \boxed{\begin{aligned} & E\left[E\left[\frac{Y_T Z_0}{Z_T Y_0}\,\middle|\,\mathcal{F}_0\right]\right] \\ &= E\left[\frac{Z_0}{Y_0} E\left[\frac{Y_T}{Z_T}\,\middle|\,\mathcal{F}_0\right]\right] \\ &= 1 \end{aligned}}$

Define $\mathbb{Q}^Y$ on $(\Omega, \mathcal{F})$ by

$$\mathbb{Q}^Y(A) = E^{\mathbb{Q}^Z}\left[I_A \frac{Y_T Z_0}{Z_T Y_0}\right]$$

Then $\mathbb{Q}^Y$ is a prob. measure $\approx \mathbb{Q}^0 \approx \mathbb{Q}^Z$.

For $0 \le s \le t \le T$,

$$E^{\mathbb{Q}^Y}\left[\frac{S_t}{Y_t}\,\middle|\,\mathcal{F}_s\right] \overset{\text{Bayes}}{=} \frac{E^{\mathbb{Q}^Z}\left[\frac{S_t}{Y_t}\frac{Y_T Z_0}{Z_T Y_0}\,\middle|\,\mathcal{F}_s\right]}{E^{\mathbb{Q}^Z}\left[\frac{Y_T Z_0}{Z_T Y_0}\,\middle|\,\mathcal{F}_s\right]}$$

condi.

$$= \frac{E^{\mathbb{Q}^Z}\left[\frac{S_t}{Y_t} E^{\mathbb{Q}^Z}\left[\frac{Y_T Z_0}{Z_T Y_0}\,\middle|\,\mathcal{F}_t\right]\,\middle|\,\mathcal{F}_s\right]}{E^{\mathbb{Q}^Z}\left[\frac{Y_T Z_0}{Z_T Y_0}\,\middle|\,\mathcal{F}_s\right]}$$

(#4)

$\{Y_t/Z_t\}$ - $\mathbb{Q}^Z$-martingale

$$= \frac{E^{\mathbb{Q}^Z}\left[\frac{S_t}{Y_t}\frac{Y_t}{Z_t}\frac{Z_0}{Y_0}\,\middle|\,\mathcal{F}_s\right]}{E^{\mathbb{Q}^Z}\left[\frac{Y_T Z_0}{Z_T Y_0}\,\middle|\,\mathcal{F}_s\right]}$$

$\{Y_t/Z_t\}$, $\{S_t/Z_t\}$ - $\mathbb{Q}^Z$ martingale

$$= \frac{S_s/Z_s}{Y_s/Z_s} = \frac{S_s}{Y_s} \text{ a.s. } \blacksquare$$

- Rewriting pricing formula

Recall ~~that~~ the price formula

$$\Pi_t = E^{\mathbb{Q}}\left[D(t,T) H \,\middle|\, \mathcal{F}_t\right].$$

If $\{Z_t\}$ is another numéraire satisfying $E^{\mathbb{Q}}[Z_T D(0,T)] = E[Z_0]$.

Then from Theorem 3.3, $\mathbb{Q}^Z$ defined by $d\mathbb{Q}^Z = \frac{Z_T B(0)}{B(T) Z_0} d\mathbb{Q}$ is a martingale measure.

(#5)

Hence under $\mathbb{Q}^Z$, the process $\left\{\frac{V_t(\phi)}{Z_t} \,\middle|\, 0 \le t \le T\right\}$ is a martingale, where $V(\phi)$ is the replication portfolio of H.

$$\therefore \frac{V_t(\phi)}{Z_t} = E^{\mathbb{Q}^Z}\left[\frac{H}{Z_T}\,\middle|\,\mathcal{F}_t\right]$$

$$\Rightarrow \Pi_t = Z_t E^{\mathbb{Q}^Z}\left[\frac{H}{Z_T}\,\middle|\,\mathcal{F}_t\right].$$

In particular if $Z_T = 1$

then $\Pi_t = Z_t E^{\mathbb{Q}^Z}[H \mid \mathcal{F}_t]$.

Does there is any choice of such a numeraire?

(#6)

Bond Market Model

On $(\Omega, \mathcal{F}, \mathbb{Q}_0)$ with $\{\mathcal{F}_t\}$ a filtration satisfying usual conditions.

Market: consists of zero coupon bonds of all maturities upto $T^* < \infty$.

The price processes of the bonds satisfy

(i) $P(t,T) = A_t^T + M_t^T$, where $\{A_t^T\}$ continuous adapted BV process, $\{M_t^T\}$ continuous square integrable martingales.

(ii) $P(t,T) \ge 0$, $P(T,T) = 1$

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(iii) There exists a probability measure $Q \approx Q_0$ such that $\left\{ \frac{P(t,T)}{B(t)} \mid 0 \le t \le T \right\}$ is a Q-martingale.

~~Lemma 4.1 For the market satisfying~~

Def (Admissible strategies)
A process $\varphi = \{(\varphi_t^1, \varphi_t^2, \dots)\}, 0 \le t \le T^*\}$ together with $\{T_1, T_2, \dots\}$ is admissible if
(i) φ is locally bounded and predictable
(ii) For each $T > 0$, $\exists\, n(T) \in \mathbb{Z}_+$ such that $\varphi_t^i = 0 \;\forall\, 0 \le t \le T, i > n(T)$ and $\varphi_t^i = 0 \;\forall\, t > T_i$
(iii) For each $T > 0$, the value process
$$V_t(\varphi) = \sum_{i=0}^{\infty} \varphi_t^i P(t, T_i), \; 0 \le t \le T$$
satisfies
$$dV_t(\varphi) = \sum_{i=0}^{\infty} \varphi_t^i \, dP(t, T_i)$$
(iv) The process $\left\{ \frac{\cancel{V_t(\varphi)}}{\cancel{B(t)}} \mid 0 \le t \le T \right\}$ $\sum_{i=0}^{\infty} \int_0^t \varphi_s^i \, d(D(0,s) P(s,T))$ is a Q-martingale.

Theorem 4.1 Let φ be admissible. Then $\left\{ \frac{V_t(\varphi)}{B(t)} \mid 0 \le t \le T \right\}$ is a Q-martingale.

(9)

Proof: For $T > 0$
$$V_t(\varphi) = \sum_{i=1}^{\infty} \varphi_t^i P(t, T_i), \; 0 \le t \le T.$$
$$d\left(\frac{V_t(\varphi)}{B(t)} \right) = \cancel{\sum_{i=1}^{\infty} d\left(\varphi_t^i \frac{P(t,T)}{B(t)} \right)}$$
$$= \cancel{\sum_{i=1}^{\infty}} = V_t(\varphi)\, d\left(\frac{1}{B(t)} \right) + \frac{1}{B(t)} dV_t(\varphi)$$
$$= \sum_{i=1}^{\infty} d\left(\frac{1}{B(t)} \right) \varphi_t^i P(t, T_i) + \sum_{i=1}^{\infty} \frac{1}{B(t)} \varphi_t^i \, dP(t, T_i)$$
$$= \sum_{i=1}^{\infty} \varphi_t^i \, d\left(\frac{P(t,T_i)}{B(t)} \right)$$
Hence from the definition it follows ~~that~~ the result.

(10)

~~Theorem 4.2~~
Def (Numéraire). A +ve process Z is said to be a numéraire if $t \ge 0$. $Z_t = V_t(\varphi)$, for some φ admissible

Def (Z, Q^Z) is said to be a numéraire pair if (i) Z is a numéraire
(ii) Q^Z is a prob measure $\approx Q_0$ and $\{ Z_t^{-1} P(t,T) \mid 0 \le t \le T \}$ is a Q^Z-martingale.

(11)

Now the following results can be proved analogously.

Theorem 4.2 : Suppose (Z, Q^Z) is a numéraire pair. If Y be another numéraire, then there exists a $Q^Y \approx Q^Z$ such that (Y, Q^Y) is a numéraire pair

proof is exactly same as the proof of Theorem 3.3.

Def: A T-claim H is said to be attainable if there exists an admissible strategy

(12)

φ such that $V_T(\varphi) = H$.

Theorem 4.3 If H is an attainable claim, then price of H at t is given by

$$\pi_t = E^{Q}\left[D(t,T)H \mid \mathcal{F}_t\right] \quad (1)$$

mimick the proof earlier.

Def: (T-forward measure)
The probability measure Q^T associated with the numéraire $Z_t = P(t,T)$ is said to be the T-forward measure.

(13)

Pricing using T-forward measure

Q^T is given by

$$Q^T(A) = E^{Q}\left[I_A \frac{Z_T B(0)}{B(T) Z_0}\right]$$

$$= E^{Q}\left[I_A \frac{1}{B(T) P(0,T)}\right]$$

Under Q^T, the process

$$\left\{ \frac{V_t(\varphi)}{P(t,T)} \;\middle|\; 0 \le t \le T \right\}$$ is

a martingale. (Use Theorem 4.2)

Since $V_T(\varphi) = H$, we have

$$\frac{V_t(\varphi)}{P(t,T)} = E^{Q^T}\left[\frac{H}{P(T,T)} \;\middle|\; \mathcal{F}_t\right]$$

(14)

$\therefore$

$$\pi_t = P(t,T)\, E^{Q^T}\left[H \mid \mathcal{F}_t\right] \quad (2)$$

Applications

- $P(t,T) = E^{Q}\left[D(t,T) \mid \mathcal{F}_t\right]$.

Take $H = 1$ and combine (1) and (2)

- T-European call option written on S-bond.

Pay off at S

$= \max\{P(T,S) - K, 0\}$.

$\therefore$ Call(t,T,S,K)

$= P(t,T) E^{T}\left[(P(T,S) - K)^+ \mid \mathcal{F}_t\right]$.

(15)

Caps & Floors

A cap is a contract where the holder has a payoff of $N\tau_i(L(T_{i-1},T_i)-K)^+$ at T_i, $i=1,2,\dots,n$

Cap can be viewed as a combination of caplets, with caplet-i paying
$$N\tau_i(L(T_{i-1},T_i)-K)^+ \text{ at } T_i$$

~~Calpe~~ Caplet price

$$\text{Caplet}(t,T_{i-1},T_i,N,K)$$
$$= E^{\mathbb{Q}}[D(t,T_{i-1})N\tau_i(L(T_{i-1},T_i)-K)^+|\mathcal{F}_t]$$

(16)

$$= E^{\mathbb{Q}}[D(t,T_{i-1})N\tau_i(L(T_{i-1},T_i)-K)^+ \, E^{\mathbb{Q}}[D(T_{i-1},T_i)|\mathcal{F}_{T_{i-1}}]|\mathcal{F}_t]$$

$$= E^{\mathbb{Q}}[D(t,T_{i-1})N\tau_i(L(T_{i-1},T_i)-K)^+ \, P(T_{i-1},T_i)|\mathcal{F}_t]$$

Now use
$$L(t,T)=\frac{1-P(t,T)}{\tau(t,T)P(t,T)}$$
we get

$$= E^{\mathbb{Q}}[D(t,T_{i-1})N[1-(1+\tau_iK)P(T_{i-1},T_i)]^+|\mathcal{F}_t]$$

$$= N(1+\tau_iK) \times E^{\mathbb{Q}}\left[D(t,T_{i-1})\left(\frac{1}{1+\tau_iK}-P(T_{i-1},T_i)\right)^+\Big|\mathcal{F}_t\right]$$

(17)

$$= N_i\,\text{Put}(t,T_{i-1},T_i,K_i),$$
where
$$N_i = N(1+\tau_iK)$$
$$K_i=\frac{1}{1+\tau_iK}.$$

$$\therefore \text{Cap}(t,K,\{T_1,\dots,T_n\},N,K)$$
$$=\sum_{i=1}^{n} N_i\,\text{Put}(t,T_{i-1},T_i,K_i)$$

(18)

Short rate models ① Lecture 5

- Input short rate dynamics $\{r(t) \mid t \geq 0\}$ under Q-measure.
- Construct Bond Market model using

$$P(t,T) \overset{def}{=} E^Q[D(t,T) \mid \mathcal{F}_t] \quad (1)$$

- Does the above market satisfy (i)–(iii).

Clearly $P(t,T) \geq 0$, $P(T,T) = 1$ hence (ii) holds.

Note
$$\frac{P(t,T)}{B(t)} = E^Q\left[\frac{D(t,T)}{B(t)} \mid \mathcal{F}_t\right] = E^Q\left[\frac{1}{B(T)} \mid \mathcal{F}_t\right]$$

Hence $\left\{\frac{P(t,T)}{B(t)} \mid 0 \leq t \leq T\right\}$ is a Q-martingale. i.e. (iii) holds.

To see whether (i) holds one need to answer the question

②

what is the dynamics of $P(t,T)$ under Q_0, the objective prob. measure.

Lemma: Let $r(\cdot)$ be given by

$$dr(t) = \mu(t, r(t))\,dt + \sigma(t, r(t))\,dW_t^0$$

where $W^0(\cdot)$ an $\mathbb{R}$-Wiener process under Q_0.

If μ, σ are jointly continuous and Lip continuous in the 2nd argument uniformly over the first, the bond price has the dynamics given by

$$\frac{dP(t,T)}{P(t,T)} = \mu^T(t, r(t))\,dt + \sigma^T(t, r(t))\,dW_t^0,$$

where μ^T, σ^T satisfies

$$\frac{\mu^T(t, r(t)) - r(t)}{\sigma^T} = \lambda(t), \quad t \geq 0,\ T > 0.$$

③

Proof ~~For~~ $T > 0$.

Set $F^T(t,x) = E^Q[D(t,T) \mid r_t = x]$.

Then one see that $F^T \in C^{1,2}([0,T] \times \mathbb{R})$ | RHS can be identified as the sol. to a PDE

and $P(t,T) = F^T(t, r(t))$.

Itô $\Rightarrow$

$$dF^T = \left(F_t^T + \mu F_x^T + \frac{1}{2}\sigma^2 F_{xx}\right)dt + \sigma F_x\,dW_t^0 \quad \ldots \quad (2)$$

choose a self-financing portfolio $\varphi = \{(\varphi_t^T, \varphi_t^S) \mid t \geq 0\}$ of the T, S-bonds.

Then value process

$$V_t(\varphi) = \varphi_t^T F^T(t, r(t)) + \varphi_t^S F^S(t, r(t))$$

④

satisfies

$$dV_t(\varphi) = \varphi_t^T\,dF^T + \varphi_t^S\,dF^S$$

$$\Rightarrow \frac{dV_t(\varphi)}{V_t(\varphi)} = u_t^T \frac{dF^T}{F^T} + u_t^S \frac{dF^S}{F^S},$$

where $u_t^T = \frac{\varphi_t^F F^T}{V_t(\varphi)}$, $u_t^S = \frac{\varphi_t^S F^S}{V_t(\varphi)}$.

Then using (2), we get

$$dV_t(\varphi) = V_t(\varphi)\left[u_t^T \mu^T + u_t^S \mu^S\right]dt + V_t(\varphi)\left[u_t^T \sigma^T + u_t^S \sigma^S\right]dW$$

where

$$\left.\begin{aligned} \mu^T(t,x) &= \frac{\left(F_t^T + \mu F_x^T + \frac{1}{2}\sigma^2 F_{xx}^T\right)(t,x)}{F^T(t,x)} \\ \sigma^T(t,x) &= \frac{(\sigma F_x^T)(t,x)}{F^T(t,x)} \end{aligned}\right\} \quad (3)$$

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choose $\varphi = \{(\varphi_t^T, \varphi_t^S)\}$ such that

$$u_t^T \sigma^T + u_t^S \sigma^S = 0,$$

since $u_t^T + u_t^S = 1$, we get

$$u_t^T(x) = \frac{\sigma^S(t,x)}{\sigma^T - \sigma^S}, \quad u_t^S(x) = \frac{\sigma^T}{\sigma^T - \sigma^S}(t,x)$$

For the above choice $V_t(\varphi)$ satisfies

$$dV_t(\varphi) = \left[\frac{\mu^S\sigma^T - \sigma^S\mu^T}{\sigma^T - \sigma^S}\right](t, r(t))\,dt$$

Using the no arbitrageness of the Market, we have

$$\frac{\mu^S\sigma^T - \sigma^S\mu^T}{\sigma^T - \sigma^S}(t, r(t)) = r(t)$$

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Rewriting this =>

$$\frac{\mu^S - r(t)}{\sigma^S}(t, r(t)) = \frac{\mu^T - r(t)}{\sigma^T} = \lambda(t)$$

Also from (2), (3) it follows that

$$dP(t,T) = F^T\mu^T dt + F^T\sigma^T dW_t^0$$

i.e.

$$\frac{dP(t,T)}{P(t,T)} = \mu^T(t, r(t))\,dt + \sigma^T(t, r(t))\,dW_t^0.$$

i.e. <u>under Q^0</u>, $P(t,T)$ satisfies

$$\frac{dP(t,T)}{P(t,T)} = (r(t) + \lambda(t)\sigma^T(t, r(t)))\,dt + \sigma^T(t, r(t))\,dW_t^0, \quad (4)$$

for some σ^T, $\lambda(\cdot)$.

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<u>given $\lambda(\cdot)$ what is Q?</u>

under Q, $\left\{\frac{P(t,T)}{B(t)}\right\}$ is a martingale.

$\therefore$ under Q, $P(t,T)$ is given by

$$dP(t,T) = r(t)P(t,T)\,dt + \sigma^T(t, r(t))P(t,T)\,dW_t \quad (5)$$

(Since Girsanov doesn't change diffusion term)

Hence Girsanov's Thm (4), (5) =>

$$\left.\frac{dQ}{dQ_0}\right|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\int_0^t \lambda^2(s)\,ds - \int_0^t \lambda(s)\,dW\right)$$

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Recall Girsanov.

under Q defined above

$$dW_t = dW_t^0 + \lambda(t)\,dt$$

is a B.M.

Hence the dynamics of $r(\cdot)$ under Q is

$$dr(t) = [\mu(t, r(t)) - \lambda(t)\sigma(t, r(t))]\,dt + \sigma(t, r(t))\,dW_t.$$

<u>Remark</u>

- $\lambda(\cdot)$ is called the market price of risk
- $\lambda(\cdot)$ decides Q
- Market tells what is $\lambda(\cdot)$
- Market uniquely pick a risk-neutral measure.

9

An example (Vasicek Model)

Under Q-dynamics $r(\cdot)$ is given by

$$dr(t) = k[\theta - r(t)]dt + \sigma dW_t,$$

$k, \theta, \sigma > 0$.

Set $X_t = e^{kt} r_t$.

Product formula $\Rightarrow$

$$dX_t = k\theta e^{kt} dt + \sigma e^{kt} dW_t$$

$$\Rightarrow X_t = X_s + \theta(e^{kt} - e^{ks}) + \sigma \int_s^t e^{ku} dW_u$$

$$\therefore r_t = r_s e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) + \sigma \int_s^t e^{-k(t-u)} dW_u \quad (6)$$

From (6) it follows that conditional on $\mathcal{F}_s$ $r(t)$ is

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$$N\left(r(s) e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}),\ \frac{\sigma^2}{2k}(1 - e^{-2k(t-s)})\right)$$

Compute $P(t,T)$

$E^{Q}[e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t]$ is the unique solution to

$$F_t + k(\theta - x)F_x + \tfrac{1}{2}\sigma^2 F_{xx} = xF$$

$$F(T, x) = 1$$

[This can be seen from (3),

$$\frac{\mu^T - r(t)}{\sigma^T} = \lambda(t), \quad \lambda(t) \equiv 0 \text{ under } Q$$

OR Feymann-Kac]

Solving the PDE in the form $F^T(t,x) = A(t,T)e^{-B(t,T)x}$

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we get

$$P(t,T) = A(t,T) e^{-B(t,T)r(t)},$$

when

$$A(t,T) = e^{\left(\theta - \frac{\sigma^2}{2k^2}\right)[B(t,T) - T + t] - \frac{\sigma^2 B^2(t,T)}{4k^2}} \quad (7)$$

$$B(t,T) = \frac{1}{k}\left[1 - e^{-k(T-t)}\right].$$

dynamic under T-forward measure

Recall that

$$Q^Y(A) = E^{Q^Z}\left[I_A \frac{Y_T Z_0}{Z_T Y_0}\right]$$

and $\left\{\frac{Y_t}{Z_t}\right\}$ is a Q^Z-martingale.

Hence for $A \in \mathcal{F}_t$

$$Q^Y(A) = E^{Q^Z}\left[I_A \frac{Y_t}{Z_t}\frac{Z_0}{Y_0}\right]$$

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OR $$\left.\frac{dQ^Y}{dQ^Z}\right|_{\mathcal{F}_t} = \frac{Y_t Z_0}{Z_t Y_0}.$$

Hence for the T-forward measure

$$\left.\frac{dQ^T}{dQ}\right|_{\mathcal{F}_t} = \frac{P(t,T)}{P(0,T)B(t)} \quad (8)$$

From (7), we have

$$dP(t,T) = A(t,T)e^{-B(t,T)r(t)}(-B(t,T))dr(t) + (\cdots)dt$$

$$= -B(t,T)P(t,T)dr(t) + (\cdots)dt$$

$$= -\sigma B(t,T)P(t,T)dW_t + (\cdots)dt$$

Hence under Q

$$dP(t,T) = r(t)P(t,T)dt - \sigma B(t,T)P(t,T)dW_t \quad (9)$$

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under Q

$$dr(t) = \mu(t, r(t))\,dt + \sigma\, dW_t,$$

when $\mu(t, x) = k\theta - kx$

and ~~dr~~ def under Q^T

$$dr(t) = \mu_T(t, r(t))\,dt + \sigma\, dW_t^T$$

[Note Girsanov doesn't change diffusion]

$\therefore$ Girsanov's Thm implies that ~~under~~ Q^T given by

$$\frac{dQ^T}{dQ} = Z_T, \quad \text{when } \{Z_t\}$$

satisfies

$$dZ_t = Z_t \left(\frac{\mu_T(t, r(t)) - \mu(t, r(t))}{\sigma} \right) dW_t \quad (10)$$

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i.e under Q_T $r(\cdot)$ given by

$$dr_t = (k\theta - kr(t) - \sigma^2 B(t,T))\,dt + \sigma\, dW_t^T$$

Girsanov's Thm

$W(\cdot)$ be a B.M on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ and $\theta(\cdot)$ is predictable process

if $Z_t = e^{\int_0^t \theta(s)\,dW_s - \frac{1}{2}\int_0^t |\theta(s)|^2 ds}$

(OR $dZ_t = Z_t\, \theta(t)\, dt$)

~~then~~ is an $\{\mathcal{F}_t\}$ martingale.

Then

$$\tilde{W}_t = W_t - \int_0^t \theta(s)\, ds$$

a B.M under $\tilde{P}$ given by

$$\frac{d\tilde{P}}{dP} = Z_T \quad [\tilde{P}(A) = E^P[\mathbb{1}_A Z_T]]$$

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From (8)

$$Z_t = \frac{1}{P(0,T)} \frac{P(t,T)}{B(t)}$$

Hence (under Q)

$$dZ_t = \frac{1}{P(0,T)}\, d\left(\frac{P(t,T)}{B(t)}\right)$$

$$= \frac{-\sigma B(t,T)}{P(0,T) B(t)} P(t,T)\, dW_t$$

(see eq. (9) & (10))

$$\therefore dZ_t = -\sigma Z_t B(t,T)\, dW_t. \quad (11)$$

From (10) & (11) we have

$$-\sigma B(t,T) = \frac{\mu_T - \mu}{\sigma}$$

$$\therefore \mu_T = \mu - \sigma^2 B(t,T) \quad (12)$$

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CIR model

under Q $r(\cdot)$ is given by

$$dr(t) = k(\theta - r(t))\,dt + \sigma\sqrt{x(t)}\, dW_t, \quad (1)$$

$k, \theta, \sigma > 0$ and $2k\theta > \sigma^2$.

(model preserves +vevity)

Under Q_0

$$dr(t) = \big(k(\theta - r(t)) + \lambda(t)\sigma\sqrt{x(t)}\big)\,dt + \sigma\sqrt{x(t)}\, dW_t^0. \quad (2)$$

Solving the PDE

$$F_t^T + k(\theta - x) F_x^T + \tfrac{1}{2}\sigma^2 x F_{xx}^T = x F^T$$

$$F^T(T, x) = 1$$

implies.

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$$P(t,T) = A(t,T)\, e^{-B(t,T) r(t)} \quad (3)$$

where $A(t,T)$, $B(t,T)$ are given by

$$\mathring{B}_t(t,T) - k B(t,T) - \tfrac{1}{2}\sigma^2 B(t,T) + 1 = 0$$

$$B(T,T) = 0$$

$$\dot{A}(t,T) = k\theta A(t,T) B(t,T)$$

$$A(T,T) = 1.$$

Same analysis done for Vasicek implies that (see (12)) under Q^T

$$dr(t) = [k\theta - (k + B(t,T)\sigma^2) r(t)]dt + \sigma\sqrt{r(t)}\, dW^T(t).$$

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Forward Rate dynamics implied by short rate

Lemma: The process $\{F(t;T,S) \mid 0 \le t \le T\}$ is a martingale under S-forward measure.

~~Proof~~.

In particular

$$F(t;T,S) = E^S[L(T,S)/\mathcal{F}_t]$$

Proof.

Recall that

$$F(t;T,S) = \frac{1}{\tau(T,S)}\left[\frac{P(t,T)}{P(t,S)} - 1\right].$$

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Hence

$$F(t;T,S)\, P(t,S) = \frac{P(t,T) - P(t,S)}{\tau(T,S)}$$

$$= V_t(\varphi)$$

when

$$\varphi_t = (\varphi_t^T, \varphi_t^S) = \left(\frac{1}{\tau(T,S)}, \frac{-1}{\tau(T,S)}\right),$$

$$0 \le t \le T.$$

$\therefore \varphi$ is admissible.

Hence

$$\left\{ F(t;T,S) = \frac{V_t(\varphi)}{P(t,S)} \,\Big|\, 0 \le t \le T \right\}$$

is a Q-martingale under Q^S.

The second part follows, since

$$F(T;T,S) = L(T,S).$$

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Under Q From (3) it follows that

$$dP(t,T) = r(t) P(t,T)\, dt - \sigma\sqrt{r(t)}\, B(t,T) P(t,T) dW_t \quad (3)$$

Hence under Q^S

$$dP(t,T) = -\sigma\sqrt{r(t)}\, B(t,T) P(t,T) dW_t^S + (\cdots) dt \quad (4)$$

Under Q^S

$$dP(t,S) = -\sigma\sqrt{r(t)}\, B(t,S) P(t,S)\, dW_t^S + (\cdots)\, dt \quad (5)$$

Now

$$dF(t;T,S) = \frac{1}{\tau(T,S)}\, d\left(\frac{P(t,T)}{P(t,S)}\right)$$

$\cancel{dF(t;T,S) =}$

(21)

$$dF(t;T,S) = \frac{1}{\tau(T,S)}\left[\frac{dP(t,T)}{P(t,S)} - \frac{P(t,T)}{P(t,S)^2}\,dP(t,S) + (\cdots)dt\right]$$

$$= \frac{1}{\tau(T,S)}\left[\big(B(t,S) - B(t,T)\big)\frac{P(t,T)}{P(t,S)} \times \sigma\sqrt{r(t)}\,dW_t^S + (\cdots)dt\right]$$

Hence Lemma $\Rightarrow$

$$dF(t;T,S) = \frac{1}{\tau(T,S)}\big(B(t,S)-B(t,T)\big) \times \frac{P(t,T)}{P(t,S)}\,\sigma\sqrt{r(t)}\,dW_t^S.$$

(22)

Now using

$$\frac{P(t,T)}{P(t,S)} = 1 + \tau(T,S)\,F(t;T,S)$$

and

$$\sqrt{r(t)} = \sqrt{\big(B(t,S) - B(t,T)\big)^{-1} \times \ln\left[1 + \tau(T,S)\,F(t;T,S)\right] \times \frac{A(t,S)}{A(t,T)}}$$

we get

$$dF(t;T,S) = \sigma\left(F(t;T,S) + \frac{1}{\tau(T,S)}\right) \times \sqrt{\big(B(t,S)-B(t,T)\big)^{-1} \times \ln\left[1+\tau(T,S)\,F(t;T,S)\right] \times \frac{A(t,S)}{A(t,T)}} \times dW_t^S.$$

(23)

Market Models

Fix some notations.

$T_0 < T_1 < \cdots < T_N$

$\tau_i = \tau(T_{i-1}, T_i)$

$P_i(t) = P(t, T_i)$

$F_i(t) = F(t; T_{i-1}, T_i)$

Recall $F_i(T_{i-1}) = L(T_{i-1}, T_i)$

and that

$\{F_i(t) \mid 0 \le t \le T_{i-1}\}$ is a

T_i – ~~Other~~ Forward measure martingale

(24)

So if under T_i-forward measure $F_i(t)$ is given by

$$dF_i(t) = \sigma_i\, F_i(t)\, dW_t^i \quad (1)$$

$(W^{T_i} \overset{\text{def}}{=} W^i)$. Then

caplet(t, T_{i-1}, T_i, K)

Recall

$$= P(t,T_i)\, E^{T_i}\left[\tau_i \left(F_i(T_{i-1}) - K\right)^+ \middle| \mathcal{F}_t\right] \quad \cdots (2)$$

From (1), we get

$$F_i(T_{i-1}) = F_i(t) \times \exp\left(-\tfrac{1}{2}\sigma_i^2 (T_{i-1} - t) + \sigma_i\big(W^i(T_{i-1}) - W^i(t)\big)\right) \quad \cdots (3)$$

(25)

Hence (Nominal 1)

$\text{Caplet}(t, T_{i-1}, T_i, K)$

$$= \tau_i P_i(t)\left[F_i(t) N(d_1^i) - K N(d_2^i)\right],$$

when

$$d_1^i = \frac{1}{\sigma_i\sqrt{T_{i-1}-t}}\left[\ln\frac{F_i(t)}{K} + \frac{1}{2}\sigma_i^2(T_{i-1}-t)\right],$$

$$d_2^i = d_1^i - \sqrt{T_{i-1}-t}$$

called the Black's formula for caplet-i denoted by $\text{Caplet}_i^B(t)$.

<u>Implied</u> <u>Black</u> <u>Volatilities</u>

Let Cap_i^m - denote the market price of cap with settlement date $T_0, T_1, .., T_i$

(26)

Then

$$\text{Caplet}_i^m(t) = \text{Cap}_i^m(t) - \text{Cap}_{i-1}^m(t),$$

$i = 1, 2, .., n$

with $\text{Cap}_0^m(t) = 0$

Black implied volatilities $\hat{\sigma}_1, .., \hat{\sigma}_n$ are defined by

$$\text{Caplet}_i^m(t) = \text{Caplet}_i^B(t, \hat{\sigma}_i).$$

<u>Def</u> (LIBOR Market Model)

LIBOR market model with resettlement dates $T_0, T_1, .., T_n$ is defined as a model for the forward rates such that forward rate $F_i(t)$

(27)

is lognormal under T_i-forward measure for all $i = 1, 2, .., n$.

<u>Question</u> Does there exist a LIBOR Market model?

Theorem: Let $\sigma_1, .., \sigma_n$ be bounded continuous functions and W^n be a Wiener process on $(\Omega, \mathcal{F}, Q^n)$. Define $\{F_i(t) \mid 0 \le t \le T_{i-1}\}$, $i = 1, 2, .., n$ as follows

$$dF_i(t) = -F_i(t)\left[\sum_{k=i+1}^{n}\frac{\tau_k F_k(t)}{1+\tau_k F_k(t)}\sigma_i(t)\sigma_k(t)\right]dt$$

$$+ F_i(t)\sigma_i(t)\,dW^n(t).$$

(28)

Then if

$$E^{Q}\left[\frac{dQ^i}{dQ^{i+1}}\Big|\mathcal{F}_t\right] = \frac{P_{i+1}(0)\,P_{i}(t)}{P_{i}(0)\,P_{i+1}(t)},$$

$i = n-1, n-2, .., 1$.

Then under Q^i

$$dF^i(t) = \sigma_i(t) F_i(t)\, dW^i(t).$$

Hence defines a LIBOR Market model.

<u>Proof</u>

The SDE has a unique solution. To see this. For $i = n$

$$dF_n(t) = F_n(t)\sigma_n(t)\,dW_t^n.$$

This clearly has a unique sol. Now for $i = n-1$

$$dF_{n-1}(t) = -F_{n-1}(t)\frac{\tau_n F_n(t)}{1+\tau_n F_n(t)}\sigma_{n-1}(t)\sigma_n(t)\,dt$$

$$+ F_{n-1}\sigma_{n-1}\,dW^n.$$

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This again has a unique sol. and proceed.

Define iteratively the following probability mean

$$E^{Q_i}\left[\frac{dQ^i}{dQ_{i+1}} \,\Big|\, \mathcal{F}_t\right] = \frac{P_{i+1}(0)\, P_i(t)}{P_i(0)\, P_{i+1}(t)},$$

$0 \le t \le T_i,\ i = n-1, \dots, 1$

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i.e Q^{n-1} is T_{n-1} - forward measure.

Similarly others.

Dynamics of $F_n(\cdot)$ under Q^n is clearly

$$dF_n(t) = \sigma_n(t) F_n(t)\, dW^n_t$$

i.e. log normal.

Since $\dfrac{dQ^n}{dQ} = \dfrac{P_n(t)}{P_n(0)\, D(0,t)}$ on $\mathcal{F}_t$

and $\dfrac{dQ^{n-1}}{dQ^n} = \dfrac{P_n(0)\, P_{n-1}(t)}{P_{n-1}(0)\, P_n(t)}$

It follows that

$$\frac{dQ^{n-1}}{dQ} = \frac{P_{n-1}(t)}{P_{n-1}(0)\, D(0,t)} \text{ on } \mathcal{F}_t$$

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set

$$\eta(t) = \frac{P_n(0)\, P_{n-1}(t)}{P_{n-1}(0)\, P_n(t)} = \frac{P_n(0)}{P_{n-1}(0)}\left(1 + \tau_n F_n(t)\right)$$

$$\therefore d\eta(t) = \frac{P_n(0)}{P_{n-1}(0)}\, \tau_n\, dF_n(t) = \frac{P_n(0)}{P_{n-1}(0)}\, \tau_n F_n(t) \sigma_n(t)\, dW^n(t)$$

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$$\therefore d\eta(t) = \eta(t)\left(\frac{\tau_n F_n(t)}{1 + \tau_n F_n(t)}\right)\sigma_n(t)\, dW^n_t$$

Solve this we set

$$\eta(t) = \exp\left(\int_0^t q(s)\, dW^n(s) - \frac{1}{2}\int_0^t |q(s)|^2 ds\right),$$

where $\quad q(t) = \dfrac{\tau_n F_n(t)}{1 + \tau_n F_n(t)}$.

Now Girsanov $\Rightarrow$

$$dW^{n-1}(t) = dW^n_t - \frac{\tau_n F_n(t)}{1 + \tau_n F_n(t)}\, dt$$

is a BM under Q^{n-1},

$\therefore$

dF_{n-1}